JOURNAL OF APPROXIMATION THEORY **43**, 178–189 (1985)

Best L_1-Approximation of Bounded, Approximately Continuous Functions on [0, 1] by Nondecreasing Functions

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Communicated by R. Bojanic

Received December 19, 1983

Let Ω denote the closed interval [0, 1] and let bA denote the set of all bounded, approximately continuous functions on Ω. Let $f \in bA$. It is shown that f has an (essentially) unique best L_1-approximation f_1 by nondecreasing functions; f_1 is shown to be continuous. For $1 < p < \infty$, the best L_p-approximations f_p are shown to be continuous, and they are shown to converge uniformly to f_1 as $p \to 1$. A characterization of f_1 is given. It is also shown that if $f^n \in bA$, $0 \leqq n < \infty$ and f^n converges to f^0 in L_1 as $n \to \infty$, then $f_1^n \to f_1^0$ in L_1 as $n \to \infty$.

1. INTRODUCTION

Let $(X, \mathcal{A}, \mu)$ be a probability space and put $A_p = L_p(X, \mathcal{A}, \mu)$, $1 \leqq p \leqq \infty$. Let $\mathcal{B}$ be a sub sigma algebra of $\mathcal{A}$ and put $B_p = L_p(X, \mathcal{B}, \mu)$, $1 \leqq p \leqq \infty$. For $1 < p < \infty$, A_p is a uniformly convex Banach space, so $f \in A_\infty$ has a unique best L_p-approximation f_p by elements of the subspace B_p. Shintani and Ando [14] investigated best L_1-approximants. In [5] it was shown that there is a unique, well defined best best L_∞-approximation f_∞ to f and f_∞ has the Polya property: $f_\infty = \lim_{p \to \infty} f_p$. This line of investigation was continued in [1–3, 6, 8]. Generalizing, let $\mathcal{M}$ be a sub sigma lattice of $\mathcal{A}$. Then $M_p = L_p(X, \mathcal{M}, \mu)$ is a closed convex cone in A_p and f has a unique best approximation f_p in M_p, $1 < p < \infty$. A basic example is obtained by putting $X = \Omega$, $m =$ Lebesgue measure and $\mathcal{A}$ the Lebesgue measurable sets in Ω. Put $\mathcal{M} = \{\phi, \Omega, (a, 1], [a, 1], 0 < a < 1\}$; then a function g on Ω is $\mathcal{M}$-measurable if and only if it is nondecreasing. Henceforth, attention is restricted to this case. The Polya property fails [8, 7]: $\lim_{p \to \infty} f_p \doteq f_\infty$ need not exist. A slight modification of the example given in [7] will appear at the end for the sake of completeness; the function f in this example is continuous except at $x = \frac{1}{2}$, but f is approximately continuous at $x = \frac{1}{2}$ and constant on $[\frac{1}{2}, 1]$. However [9], if

0021-9045/85 $3.00

f is quasi-continuous, the Polya property obtains. In fact, $f_p \to f_\infty$ uniformly as $p \to \infty$; moreover, if f is continuous, then f_p is continuous, $1 < p < \infty$. Herein we will look at the corresponding Polya-one property: $\lim_{p \downarrow 1} f_p$, and at existence and uniqueness of best L_1-approximations to f in bA by nondecreasing functions. The results in [9] establish that f_∞ is a best L_∞-approximation to f when f is quasi-continuous. Of course, even when f is continuous there may be many best L_∞-approximations. The indicator function $I_{[0,1/2]}$ of the interval $[0, \frac{1}{2}]$ has any constant function with value between zero and one as a best L_1-approximation by elements of M_1. (These constant functions are also the best L_1-approximations to f by elements of B_1 when $\mathscr{B} = \{\phi, [0, 1]\}$.) One of the authors [10] established the Polya-one property for quasi-continuous functions. For f in bA, we will show that there is an (essentially) unique best L_1-approximation f_1 to f by nondecreasing functions, that f_1 is continuous, and that f_p converges uniformly to f_1 as $p \to 1$. We will characterize f_1. The only ambiguity in f_1 occurs at the endpoints zero and one, so uniqueness obtains if we specify that the nondecreasing approximations be continuous at zero and one.

We will also look at continuity of the map $f \to f_1$. The map $f \to f_p$ is uniformly continuous in $\|\cdot\|_p$ on bounded subsets of L_∞ for fixed p, $1 < p < \infty$. The map $f \to f_\infty = \lim_{p \to \infty} f_p$ is uniformly continuous in $\|\cdot\|_\infty$ on the quasi continuous functions. We will give an example (Example 2) to show that the map $f \to f_1$ is not uniformly continuous in $\|\cdot\|_1$ on $C[0, 1]$. But we will show that the map $f \to f_1$ is continuous in $\|\cdot\|_1$ on the bounded approximately continuous functions.

2. Existence, Uniqueness, and Two Characterizations of f_1

If A is a measurable subset of Ω and I is a subinterval of Ω, the *relative measure of A in I* is defined by

$$m(A, I) = m(A \cap I)/mI$$

and the *upper metric density of A at x*, x in Ω, is defined by

$$\bar{m}(A, x) = \lim_{n \to \infty} \sup_I \{m(A, I): I \text{ is an interval, } x \in I, \text{ and } mI < 1/n\}.$$

The *lower* metric density $\underline{m}(A, x)$ is defined similarly, with sup replaced by inf. The *metric density* of A at x is $m(A, x) = \bar{m}(A, x) = \underline{m}(A, x)$, when equality holds. A function $f: \Omega \to R$ is said to be *approximately continuous* at x in Ω if, for any $\varepsilon > 0$, the set

$$A_\varepsilon = \{y: |f(y) - f(x)| < \varepsilon\}$$

has metric density one at x; f is said to be approximately continuous on Ω if it is approximately continuous at each point in Ω.

Remark. A function f is approximately continuous at x if and only if x is a Lebesgue point of f [4, p. 38; 13, p. 168]. Reference [4] contains a nice introduction to approximately continuous functions and a rather complete set of references.

Let M consist of all functions $g: \Omega \to R$ such that g is nondecreasing, $g(0) = \inf\{g(x): x \in (0, 1)\}$ and $g(1) = \sup\{g(x): x \in (0, 1)\}$. We suppose $f \in bA$. For $1 < p < \infty$, let f_p denote the unique best L_p-approximation to f by elements of M.

LEMMA 1. *Suppose $1 \leqq p$ and g is a best L_p-approximation to f by elements of M. Then g is continuous.*

Proof. Suppose first that $g \equiv f$. Then f is nondecreasing, so f has at most discontinuities of the first kind and f is *quasi-continuous*; i.e., for any y in $(0, 1)$, f has left and right limits at y: $f(y-) = \lim_{x \uparrow y} f(x)$ and $f(y+) = \lim_{x \downarrow y} f(x)$ both exist. If $0 < y < 1$ and $f(y-) < f(y+)$, then f is not approximately continuous at y. Thus f is in fact continuous and the assertion of the lemma is true.

Suppose $g \not\equiv f$. First we will consider points y where $g(y) \neq f(y)$. We will only consider the case where $0 < y < 1$ and $f(y) - g(y) = 3\varepsilon > 0$ because the other cases are similar. Let $Q \in (0, 1)$. We will specify Q later. Since f is approximately continuous, there exists $\delta = \delta_Q > 0$ such that

$$m([f > f(y) - \varepsilon], I) > Q \tag{1}$$

for any interval I such that $y \in I$ and $I \subset B(y, \delta) = (y - \delta, y + \delta)$. We now suppose that $\eta = \min\{g(y+) - g(y-), \varepsilon\} > 0$ and show that this supposition leads to a contradiction.

Define $\phi: \Omega \to R$ by

$$\begin{aligned} \phi(x) &= g(x) + \eta, \qquad && x \in (y - \delta, y) \\ &= g(y-) + \eta, && x = y \\ &= g(x), && x \notin (y - \delta, y]. \end{aligned} \tag{2}$$

Let $I = (y - \delta, y]$ and $F = I \cap [f > f(y) - \varepsilon]$. Applying the mean value theorem to the function $s \mapsto s^p$ we have that there exists a u in $(s, s + \sigma)$ such that

$$(s + \sigma)^p - s^p = pu^{p-1}\sigma \geqq ps^{p-1}\sigma,$$

so, for t in F,

$$|f(t)-g(t)|^p - |f(t)-\phi(t)|^p \geqq p\,|f(t)-\phi(t)|^{p-1}\,|\phi(t)-g(t)|$$

whence

$$|f(t)-\phi(t)|^p \leqq |f(t)-g(t)|^p - p\,|f(t)-\phi(t)|^{p-1}\,\eta.$$

Then

$$\int_F |f-\phi|^p \leqq \int_F |f-g|^p - p\eta \int_F |f-\phi|^{p-1} \leqq \int_F |f-g|^p - p\eta\varepsilon^{p-1}\,\delta Q. \quad (3)$$

Notice also that

$$|\,|f(t)-\phi(t)|^p - |f(t)-g(t)|^p| \leqq p(2\,\|f\|_\infty)^{p-1}\,|\phi(t)-g(t)|;$$

thus,

$$\left|\int_{I-F} |f-\phi|^p - \int_{I-F} |f-g|^p\right| \leqq p(2\,\|f\|_\infty)^{p-1}\,\eta m(I-F)$$

$$< p(2\,\|f\|_\infty)^{p-1}\,\eta(1-Q)\,\delta.$$

So, if we choose $Q \in [0, 1]$ satisfying

$$\varepsilon^{p-1}Q > \|2f\|_\infty^{p-1}(1-Q), \quad (4)$$

we find that ϕ is a better L_p-approximation to f. This contradiction verifies the continuity of g at y, where $g(y) \neq f(y)$.

Suppose $g(y)=f(y)$. Then a slight variation of the above argument shows that $g(y+)-g(y)=3\varepsilon>0$ and $g(y)-g(y-)=3\varepsilon>0$ each lead to a contradiction, so Lemma 1 is established.

Before going on we wish to comment on the proof of Lemma 1. Looking first at the last two sentences of the proof, suppose $|g(y)-f(y)|<3\varepsilon$; then $g(y+)-g(y) \geqq 6\varepsilon$ and $g(y)-g(y-) \geqq 6\varepsilon$ each lead to a contradiction. Second, note that if Q_0 fits (4) for ε_0, then Q_0 fits for $\varepsilon \geqq \varepsilon_0$. Also observe that if Q_0 fits (4) for p_0 and for 1, then Q_0 fits for $p \in [1, p_0]$. We will use these comments in the following.

LEMMA 2. *Given $p>1$ and $y \in [0, 1]$, the family $\mathscr{F}_p = \{f_t : 1 < t \leqq p\}$ is equicontinuous at y.*

Proof. Referring to the proof of Lemma 1, we consider $y \in (0, 1)$. Suppose that $\mathscr{F}_p$ is not equicontinuous at y. Then there exist $\varepsilon>0$, $p_n \in (1, p]$, $|x_n-y|=\gamma_n \to 0$ with $|f_{p_n}(x_n)-f_{p_n}(y)|>8\varepsilon$. Since $\{f_{p_n}(y)\}$ is a bounded sequence, we also suppose that $|f_{p_n}(y)-\alpha|<\varepsilon$. Put $\eta=\varepsilon$ and let Q satisfy (4) for 1 and p. Then choose $\delta>0$ so that

$$m([|f-f(y)|<\varepsilon], I)>(1+Q)/2 \tag{5}$$

whenever $y\in I\subset B(y,\delta)$. The argument for $x_n<y$ is symmetric to our argument for $x_n>y$, so we suppose $x_n-y=\gamma_n>0$. Now compare $f(y)$ and α. If $f(y)\geqq\alpha+4\varepsilon$, raise f_{p_n} by ε on $(y-\delta, y)$, do not change f_{p_n} off $(y-\delta, x_n)$, and maintain monotonicity. If $f(y)<\alpha+4\varepsilon$, lower f_{p_n} by ε on $(x_n, y+\delta)$, do not change f off $(y, y+\delta)$, and maintain monotonicity. Since $\gamma_n\to 0$, $\liminf m([|f-f(y)|<\varepsilon]\cap(x_n, y+\delta),(y, y+\delta))\geqq(1+Q)/2$. Thus, a slight variation of the proof of Lemma 1 produces a contradiction as $n\to\infty$.

LEMMA 3. *Let $d_p(f, M)=\inf\{\|f-h\|_p: h\in M\}$. Then $d_p(f, M)$ is a nondecreasing function of p and*

$$\lim_{p\downarrow 1} d_p(f, M)=d_1(f, M).$$

Proof. If $0<r<s$, then, for all $h\in M$, $\|f-h\|_r\leqq\|f-h\|_s$ [13, p. 73] so $d_p(f, M)$ is a nondecreasing function of p.

It is clear that $d_p(f, M)=d_p(f, M_f)$, where $M_f=\{h\in M: \|h\|_\infty\leqq\|f\|_\infty\}$. If $d_1(f, M)=0$, then f is nondecreasing and we are done; suppose that $d_1=d_1(f, M)>0$. Let $h_n\in M_f$ with $\|f-h_n\|_1\to d_1$: $d_1=\lim_n\|f-h_n\|_1\leqq\|f-f_p\|_1\leqq\|f-f_p\|_p\leqq\liminf_n\|f-h_n\|_p$. Thus, we need to compare $\|f-h\|_p$ with $\|f-h\|_1$, where $h\in M_f$; put $\phi=|f-h|$. Then $H=\|f-h\|_\infty\leqq 2\|f\|_\infty=F$. If $H\leqq 1$, then $\int\phi^p\leqq\int\phi$ and $\|\phi\|_p\leqq(\|\phi\|_1)^{1/p}$. If $H>1$, then $\int(\phi^p-\phi)\leqq\int(H^p-H)\leqq F^p-F$ and

$$\int\phi^p\leqq(F^p-F)+\|\phi\|_1\leqq\|\phi\|_1[1+(F^p-F)/d_1].$$

Hence, $\|\phi\|_p\leqq(\|\phi\|_1)^{1/p}[1+(F^p-F)/d_1]$, $\phi=f-h$, $h\in M_f$. Thus, lim inf $\|f-h_n\|_p\leqq d_1^{1/p}[1+(F^p-F)/d_1]$ and Lemma 3 is established.

LEMMA 4. *There is a best L_1-approximation to f by elements of M.*

Proof. Referring to Lemma 2, let $\mathscr{F}$ denote the equicontinuous family $\mathscr{F}_2$. The fact that $\mathscr{F}$ is uniformly bounded allows us to apply a Theorem of Helly [12, p. 221]: If $\{p_n\}$ is any sequence with $p_n\downarrow 1$, then there is a subsequence $\{f_{q_n}\}$ of $\{f_{p_n}\}$ and a function, call it $f_1(\{q_n\})$, in M, such that $f_{q_n}\to f_1(\{q_n\})$ pointwise. Then $f_{q_n}-f\to f_1(\{q_n\})-f$ pointwise, so, by the Lebesgue Convergence Theorem,

$$\|f_{p_n}-f\|_1\to\|f_1(\{q_n\})-f\|_1.$$

Since, for every n,

$$d_1(f, M) \leqq \| f_{p_n} - f \|_1 \leqq \| f_{p_n} - f \|_{p_n},$$

Lemma 3 implies that

$$\| f_1(\{q_n\}) - f \|_1 = d_1(f, M)$$

so $f_1(\{q_n\})$ is a best L_1-approximation to f by elements of M.

Having established existence of a best L_1-approximation, we turn to uniqueness. The next lemma gives four properties that a best L_1-approximation must possess. First, a definition: a nondecreasing continuous function g on $[0, 1]$ is said to *increase to the right* at a point s of $[0, 1)$ if $x > s$ implies $g(x) > g(s)$. Similarly, g is said to increase *from the left* at t in $(0, 1]$ if $x < t$ implies $g(x) < g(t)$.

LEMMA 5. *If g is a best L_1-approximation to f by elements of M, then g is continuous and for any s in $[0, 1)$, if g increases to the right at s, then, for any t in $(s, 1]$,*

$$m\{[f<g] \cap [s, t]\} \leqq (t-s)/2; \tag{6}$$

for any t in $(0, 1]$, if g increases from the left at t, then for any s in $[0, t)$,

$$m\{[f>g] \cap [s, t]\} \leqq (t-s)/2; \tag{7}$$

for any s in $[0, 1)$,

$$m\{[f>g] \cap [s, 1]\} \leqq (1-s)/2; \tag{8}$$

and for any t in $(0, 1]$,

$$m\{[f<g] \cap [0, t]\} \leqq t/2. \tag{9}$$

Proof. If g is the best L_1-approximation to f by elements of M, then, by Lemma 1, g is continuous. Suppose condition (6) does not hold. Then there exist s and t such that $0 \leqq s < t \leqq 1$ and g increases to the right at s, but

$$m\{[f<g] \cap [s, t]\} > (t-s)/2.$$

Since $[f<g] = \bigcup_{n=1}^{\infty} [f < g - 1/n]$, there exist n' in N and $\delta > 0$ such that

$$m\{[f < g - 1/n'] \cap [s, t]\} > (t-s)/2 + 2\delta.$$

For each $n > 0$, $g - 1/n$ is continuous and increasing to the right at s and $g - 1/n \to g$ uniformly so there exist n'' in N and $x' > s$ such that

$g(x') - 1/n'' = g(s)$ and $x' - s < \delta$. Let $v = \min\{1/n', 1/n''\}$ and define θ: $[0, 1] \to R$ by

$$\begin{aligned} \theta(x) &= \min\{g(x), g(x') - v\}, && x \in [s, x'], \\ &= g(x) - v, && x \in (x', t], \\ &= g(x), && x \notin [s, t]. \end{aligned}$$

Then θ is nondecreasing and

$$\begin{aligned} \int_0^1 |\theta - f| &\leqq \int_0^s |g-f| + \int_t^1 |g-f| + \int_s^{x'} |g-f| + v\delta \\ &\quad + \int_{x'}^t |g-f| - v[(t-s)/2 + \delta] + v[(t-s)/2 - 2\delta] \\ &\leqq \int_0^1 |g-f| - 2v\delta. \end{aligned}$$

Thus, θ is a better L_1-approximation to f by elements of M than is g, a contradiction. That the other three conditions hold is proven similarly: If (7) (respectively (8), (9)) is false, we may produce a contradiction by increasing (increasing, decreasing) g on an interval of the form $[s, t-\delta)$ (respectively, $(s+\delta, 1]$, $[0, t-\delta)$).

COROLLARY 6. *Conditions* (8) *and* (9) *in Lemma* 5 *imply that* $m[f>g] \leqq \frac{1}{2}$ *and* $m[f<g] \leqq \frac{1}{2}$.

We will establish uniqueness of best L_1-approximation of f in bA by elements of M and, at the same time, characterize the best approximation by showing that only one continuous function in M satisfies the conclusion of Lemma 5.

LEMMA 7. *Let each of g and h satisfy the conclusion of Lemma* 5. *Then* $g = h$.

Proof. Suppose $g > h$. By Corollary 6, there exist y and z in $(0, 1)$ such that $f(y) \leqq h(y)$ and $f(z) \geqq g(z)$. Since $f - (g+h)/2$ is approximately continuous it has the intermediate value property, so there exist $\varepsilon, \delta > 0$ and w in $(0, 1)$ such that $h(x) < f(w) - \varepsilon < f(w) + \varepsilon < g(x)$ whenever $x \in B(w, \delta) = (w-\delta, w+\delta)$ and, since $m([|f - f(w)| < \varepsilon], B(w, \delta)) > 0$,

$$m[g > f > h] > 0. \tag{10}$$

Since $m[f > h] \leqq \frac{1}{2}$, $m[f < g] \leqq \frac{1}{2}$ and

$$\Omega = [f > g] \cup [f = g] \cup [g > f > h] \cup [f = h] \cup [f < h],$$

$m\Omega = m[f>h] + m[f<g] - m[g>f>h] < 1$, a contradiction. Thus $g>h$ on $(0, 1)$ is impossible. By symmetry, $g<h$ on $(0, 1)$ is also impossible. Suppose $g \neq h$. Then there exists u in $(0, 1)$ such that $g(u) = h(u)$ and at least one of the following three cases occurs. Case 1: $u<1$ and there exists $v \leqslant 1$ such that (u, v) is a component of $[g \neq h]$. Case 2: $u<1$ and $g(x) \neq h(x)$ for x in $(u, 1]$. Case 3: $g(x) \neq h(x)$ for x in $[0, u)$. We begin with Case 1. Suppose without loss of generality that $g>h$ on $I=(u, v)$. Then g must increase to the right at u and h must increase from the left at v; hence, according to Lemma 5, $m([f<g] \cap I) \leqslant (v-u)/2$ and $m([f>h] \cap I) \leqslant (v-u)/2$. Thus, by an argument similar to that establishing (10), $m([g>f>h] \cap I) > 0$. From the decomposition

$$I = ([f \geqslant g] \cap I) \cup ([g>f>h] \cap I) \cup ([f \leqslant h] \cap I),$$

we see that $m([f \geqslant g] \cap I) < (v-u)/2$ or $m([f \leqslant h] \cap I) < (v-u)/2$. This contradiction completes Case 1. The other cases follow by similar arguments.

We have established the following:

THEOREM 8. *Let $f \in bA$. Then there exists a unique best L_1-approximation f_1 to f by elements of M.*

Our next result shows that the best L_p-approximations f_p to f converge uniformly to f_1 as p decreases to one.

THEOREM 9. *Let $f \in bA$. Then f_p converges uniformly to f_1 as p decreases to one.*

Proof. Referring to the proof of Lemma 4, let $f_1 = f_1(\{q_n\})$. If $\{p_k\}$ is any sequence with $p_k \downarrow 1$, then $\{f_{p_k}\}$ has a subsequence which converges pointwise to f_1, the best L_1-approximation to f by elements of M. We claim that f_p converges uniformly to f_1 as p decreases to one. Indeed, if this were not true then there would be an $\varepsilon>0$ and sequences $\{p_k\} \subset (1, \infty)$ and $\{x_k\} \subset \Omega$ such that $p_k \downarrow 1$ and, for k in N,

$$|f_{p_k}(x_k) - f_1(x_k)| \geqslant \varepsilon. \tag{11}$$

By the above, $\{p_k\}$ has a subsequence $\{q_k\}$ such that $f_{q_k} \to f_1$ pointwise and, by the Ascoli–Arzela theorem, $\{q_k\}$ has a subsequence $\{r_k\}$ such that f_{r_k} converges uniformly. Clearly $f_{r_k} \to f_1$ pointwise so $f_{r_k} \to f_1$ uniformly, but this contradicts (11).

For the sake of completeness, we conclude this section with the following:

Proposition 10. *Suppose $1 \leqslant p$ and g is a best L_p-approximation to f by elements of M. If $y \in \Omega$ and $g(y) \neq f(y)$, then there exists $\delta > 0$ such that g is constant on $B(y, \delta)$.*

Proof. Suppose $y \in (0, 1)$, $g(y) \neq f(y)$, and Proposition 10 is false at y. Then either g increases to the right at y or g increases from the left at y. We will consider both cases for $f(y) - g(y) = 3\varepsilon > 0$. For the former case, in accordance with (4), let $Q \in (0, 1)$ satisfy $\varepsilon^{p-1}Q > (2 \|f\|_\infty)^{p-1}(1-Q)$. Then let $\delta = \delta_{(1+Q)/2}$ fit (1). Let $x \in (y, y + \delta(1-Q)/2)$ with $\varepsilon > \eta = f(x) - f(y) > 0$. Put

$$\begin{aligned} \phi(t) &= g(t) + \eta, && t \in (y - \delta, y], \\ &= g(x), && t \in (y, x], \\ &= g(t), && t \notin (y - \delta, x], \end{aligned}$$

and find that ϕ is a better L_p-approximation to f. (Note that $(1 - (1+Q)/2) = (1-Q)/2$ and see the argument following (2).) For the other case, let $x \in (y - \delta(1-Q)/2, y)$ with $\varepsilon > \eta = f(y) - f(x) > 0$. Then, to find a better L_p-approximation, put

$$\begin{aligned} \phi(t) &= g(t) + \eta, && t \in (y - \delta, x], \\ &= g(y), && t \in (x, y], \\ &= g(t), && t \notin (y - \delta, y]. \end{aligned}$$

When y equals zero or one a similar argument applies.

3. Continuity Properties of Best L_1-Approximations

Two examples are given here; the first shows that $\{f^n : n \geqq 1\}$ bounded in C, the set of continuous functions on $[0, 1]$, and $f^n \to f \in C$ pointwise (i.e., $f^n \to f$ weakly) does not imply that $f_1^n \to f_1$ weakly, and the second shows that $\{f^n\}$ bounded in C and pointwise convergent on $[0, 1]$ does not imply that $\{f_1^n\}$ is Cauchy in L_1. However, it is shown that if $f^n \in bA$, $0 \leqq n$, and $f^n \to f^0$ in L_1, then $f_1^n \to f_1^0$ in L_1, so the map $f \mapsto f_1$ is continuous on the elements of bA in L_1-norm. The examples show that this result does not extend in certain directions.

Example 1. Put $f^n(x) = 0$, $x \in [0, 1 - 2/n]$, $f^n(1 - 1/n) = 1$, $f^n(1) = 0$, and extend f^n to be linear on $[1 - 2/n, 1 - 1/n]$ and $[1 - 1/n, 1]$. Then $f^n \to 0$ pointwise and $f_1^n \to I_{[1]}$, where $I_E(x) = 1$ if $x \in E$ and $I_E(x) = 0$ otherwise.

EXAMPLE 2. Put $f^n(x)=1$, $x\in[0,\frac{1}{2}(1-1/n)]$, $f^n(x)=0$, $x\in[\frac{1}{2},1]$, and extend f^n to be linear on $[\frac{1}{2}(1-1/n),\frac{1}{2}]$. Put $g^n(x)=f^n(x)$, $x\in[0,\frac{1}{2}]\cup[\frac{1}{2}(1+4/n),1]$, $g^n(x)=1$, $x\in[\frac{1}{2}(1+1/n),\frac{1}{2}(1+3/n)]$, and extend g^n to be linear on each of $[\frac{1}{2},\frac{1}{2}(1+1/n)]$, $[\frac{1}{2}(1+3/n),\frac{1}{2}(1+4/n)]$. Then $f^n\leqq g^n$, $g^n(0)=f^n(0)$, $\int_0^1(g^n-f^n)\to 0$, and $g_1^n-f_1^n=g_1^n\equiv 1$. Notice that $f^n\to I_{[0,1/2)}$ pointwise and $g^n\to I_{[0,1/2)}$ pointwise. $I_{[0,1/2)}$ is quasi-continuous and has only one point of discontinuity on $[0,1]$, so the following theorem is tight.

THEOREM 11. *Let $\{f^n\}\subset bA$. Suppose $f\in bA$ with $\int_0^1|f^n-f|\to 0$. Then $\int_0^1|f_1^n-f_1|\to 0$.*

Proof. We will reduce the problem to a special case in steps. Since a subsequence of $\{f^n\}$ converges a.e. to f, suppose without loss of generality that $f^n\to f$ a.e.. We suppose that $\|f_1^n-f_1\|_1\not\to 0$ and show that this supposition leads to a contradiction below.

Our next reduction uses the inequality $\|\phi_1\|_1\leqq\|\phi\|_1+\|\phi-\phi_1\|_1\leqq 2\|\phi\|_1$, $\phi\in C$ and the convergence of $\|f^n\|_1$ to $\|f\|_1$, to assert that $\{f_1^n\}$ is uniformly bounded on $[0,x]$, $x<1$; so every subsequence of $\{f_1^n\}$ has a pointwise convergent subsequence therefrom. Consequently (without loss of generality) suppose that $f_1^n\to g$ pointwise (it is possible that $g(1)=\infty$), where $\|g-f_1\|_1>0$. If $\phi\leqq\psi\in bA$, then $\phi_p\leqq\psi_p$ for $1<p<\infty$ [11], so Theorem 9 implies that $\phi_1\leqq\psi_1$. Consequently, we consider $\{f^n\wedge f\}$ and $\{f^n\vee f\}$: $f^n\wedge f\to f$ a.e. (and in L_1) and $(f^n\wedge f)_1\leqq f_1^n\wedge f_1\to g\wedge f_1$; likewise $f^n\vee f\to f$ and $(f^n\vee f)_1\geqq f_1^n\vee f_1\to g\vee f_1$. At least one of $\int_0^1 f_1-(g\wedge f_1)$ and $\int_0^1(g\vee f_1)-f_1$ is positive and proofs for the two cases are similar, so we suppose that $f^n\geqq f$ with $f_1^n\to g\geqq f_1$ and $\int_0^1(g-f_1)>0$.

Now we have $f^n\geqq f$, $f^n\to f$ a.e., $f_1^n\to g$ pointwise and $\int_0^1(g-f_1)>0$. Let $z\in(0,1)$ with $g(z)>f_1(z)$. Since g is nondecreasing and f_1 is continuous, suppose without loss of generality that $g(z+)=g(z)$. Let $x\in[0,z]$ satisfy (i) $g(x+)=g(z)$ and (ii) $g(t)<g(z)$, $t<x$. Put $h(0)=g(0)$ and $h(t)=g(t-)$, $t>0$, so h is lower semicontinuous. Either there is a smallest number $y\in(x,1]$ with $h(y)=f_1(y)$ or there exists $\alpha>0$ with $g(t)\geqq h(t)\geqq f_1(t)+\alpha$, $t\in[x,1]$. It is easy to modify our proof for the former case to handle the latter, so we suppose that $h(y)=f_1(y)$ and $h>f_1$ on (x,y). Since h is nondecreasing, y is a point of increase of f_1 from the left: $f_1(t)<f_1(y)$, $t<y$. By Lemma 5, $m([f>f_1]\cap[x,y])\leqq(y-x)/2$. Either $x=0$ or $x>0$. A proof for the case $x=0$ follows easily from the argument given below for the case $x>0$, so we suppose that $x>0$ and verify that the promised contradiction obtains for this special case.

For the case at hand, notice that $f_1^n(x+\varepsilon)\to g(x+\varepsilon)\simeq g(x+)=g(z)$ and $f_1^n(x-\varepsilon)\to g(x-\varepsilon)<g(z)$. So (without loss of generality) we consider $x_n\to x$ such that $f_1^n(x_n)\to g(z)$ and x_n is a point of increase of f_1^n to the right: $f_1^n(t)>f_1^n(x_n)$, $t>x_n$. Thus, we can lower f_1^n on $[x_n,y)$ and maintain its nondecreasing property on $[0, 1]$ if we wish, so (cf. Lemma 5 again) $m([f^n \geqq f_1^n] \cap [x_n, y]) \geqq (y - x_n)/2$. Put $E_n = [f^n \geqq f_1^n] \cap [x_n, y]$, $F_m = \bigcup_{n \geqq m} E_n$ and $F_\infty = \bigcap_m F_m$. Then $F_\infty \subset [x, y]$ and $m(F_\infty) \geqq (y - x)/2$. Let $F = F_\infty \cap \{t: f_{(t)}^n \to f(t)\} \cap (x, y)$. Then $t \in F$ implies that $f^n(t) \to f(t)$, $f_1^n(t) \to g(t)$, and $f^n(t) \geqq f_1^n(t)$ for infinitely many positive integers: $f(t) \geqq g(t)$. Since $h - f_1$ is lower semicontinuous and positive on $[a, b]$, there exists $\varepsilon > 0$ such that $h - f_1 \geqq \varepsilon$ on $[a, b]$. Since $f - f_1$ is approximately continuous and there exist u and v in $[a, b]$ with $f(u) - f_1(u) \geqq \varepsilon$ and $f(v) - f_1(v) \leqq 0$, there must be a t in (a, b) such that $f(t) - f_1(t) = \varepsilon/2$. Consequently, $m[g > f > f_1] \geqq m[h > f > f_1] > 0$. But now we have $m([f \geqq g] \cap (x, y)) \geqq (y - x)/2$, $m([g > f > f_1] \cap (x, y)) > 0$, and $m([f \leqq f_1] \cap (x, y)) \geqq (y - x)/2$, a contradiction that establishes our desideratum.

4. An Example

In this section, we reproduce an example from [7] of a bounded function f defined on $[0, 2]$ which is continuous on $[0, 1)$, continuous on $[1, 2]$, approximately continuous at one and does not have the Polya property. A similar function $\hat{f}$ defined on $[0, 1]$ is then obtained by putting $\hat{f}(t) = f(2t)$: $\hat{f} \in bA$, $\hat{f}$ does not have the Polya property, and $\{\hat{f}_p: 1 < p < \infty\}$ is not equicontinuous.

To begin, put $f = 5$ on $[1, 2]$. We will use a sequence $\{u_n, v_n\}$ of pairs of points, $0 = u_1 < v_1 < u_2 < \cdots \to 1$, to define f on $[0, 1)$. To help explain, we record some properties that will be satisfied: $(v_n - u_n) < 4^{-n}(1 - u_n)$, $(1 - u_{n+1}) < 4^{-n}(1 - u_n)$, $f(x) = 10$ on $[u_{2n-1}, v_{2n-1}]$, $f(x) = 0$ on $[u_{2n}, v_{2n}]$, and $f(x) \neq 5$ on less than 4^{-n} percent of $[v_n, u_{n+1}]$.

The following facts will be used repeatedly. If $g \leqslant h$, then (cf. [11]) $g_p \leqq h_p$, $1 < p < \infty$, and the map $g \mapsto g_p$ is continuous in L_p, $1 < p < \infty$.

To start the definition of f on $[0, 1)$, choose $v_1 < 4^{-1}$. Put $f = 10$ on $[0, v_1]$. Proceeding in steps, we will define f temporarily on $(v_1, 1)$, modify the temporary definition ϕ of f and then define f on one more piece, $[v_1, u_2]$, of $[0, 1)$. Put $\phi = 5$ on $(v_1, 1)$ and $\phi = f$ elsewhere. Begin to increase p from one. As p increases, $\phi_p \equiv c_p$ increases from 5 to 7.5. Choose $p_1 > 1$ with $\phi_{p_1} > 7$. Put $\phi^t = \phi - 5I_{(t,1)}$, $v_1 < t < 1$. Choose u_2 with $(1 - u_2) < 4^{-1}(1 - u_1)$ and $\phi_{p_1}^{u_2} > 7$. Modify ϕ^{u_2} on less than 4^{-1} percent of $[v_1, u_2]$ so that it decreases continuously from 10 to 0 and retains the property $\phi_{p_1}^{u_2} > 7$. Put $f = \phi^{u_2}$ on $[v_1, u_2]$. Since $\phi^{u_2} = 0$ on $(u_2, 1)$, $f_{p_1} > 7$ if $f \geqq 0$ on $(u_2, 1)$.

To continue the definition of f, choose v_2 with $(v_2 - u_2) < 4^{-2}(1 - u_2)$. Put $f = 0$ on $[u_2, v_2]$. Put $\phi = 5$ on $(v_2, 1)$ and $\phi = f$ elsewhere. Begin to increase p from p_1. As p increases, ϕ_p decreases to 5 on $[0, 2]$. Choose $p_2 > 2p_1$ with $\phi_{p_2} < 6$. Put $\phi^t = \phi + 5I_{(t,1)}$, $v_2 < t < 1$. Choose u_3 with

$(1-u_3)<4^{-2}(1-u_2)$ and $\phi_{p2}^{u_3}<6$. Modify ϕ^{u_3} on less than 4^{-2} percent of $[v_2, u_3]$ so that it increases continuously from 0 to 10 and retains the property $\phi_{p2}^{u_3}<6$. Put $f=\phi^{u_3}$ on $[v_2, u_3]$. Since $\phi^{u_3}=10$ on $(u_3, 1)$, $f_{p_2}<6$ if $f \leqq 10$ on $(u_3, 1)$.

One facet of the construction remains to be displayed, so we begin one more step. Choose v_3 with $(v_3-u_3)<4^{-3}(1-u_3)$. Put $f=10$ on $[u_3, v_3]$. Put $\phi=5$ on $(v_3, 1)$ and $\phi=f$ elsewhere. Begin to increase p from p_2. As p increases, ϕ increases to 7.5 on $[u_3, 2] \supset [1, 2]$. Continuing our procedure produces a function f with the promised properties: $f_{p_{2n-1}}>7$ on $[1, 2]$ and $f_{p_{2n}}<6$ on $[1, 2]$.

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